Sampling Analysis using Correlations
for Monte Carlo Rendering

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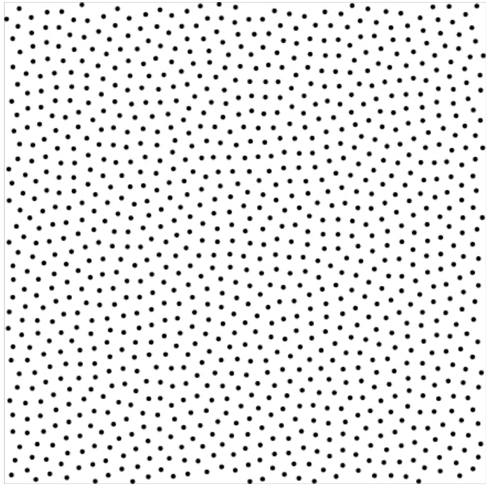
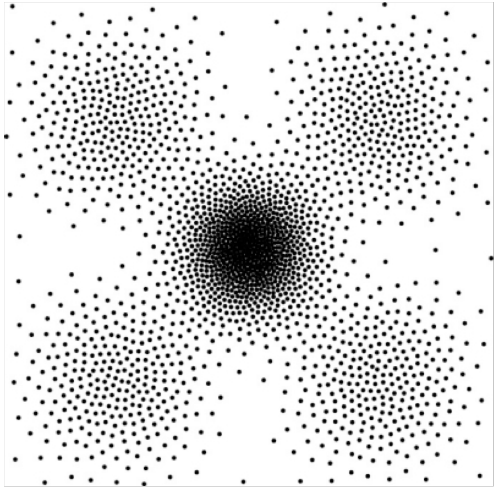
Point Patterns in Computer Graphics

Random distributions of points with characteristics

Fundamental for many applications in graphics



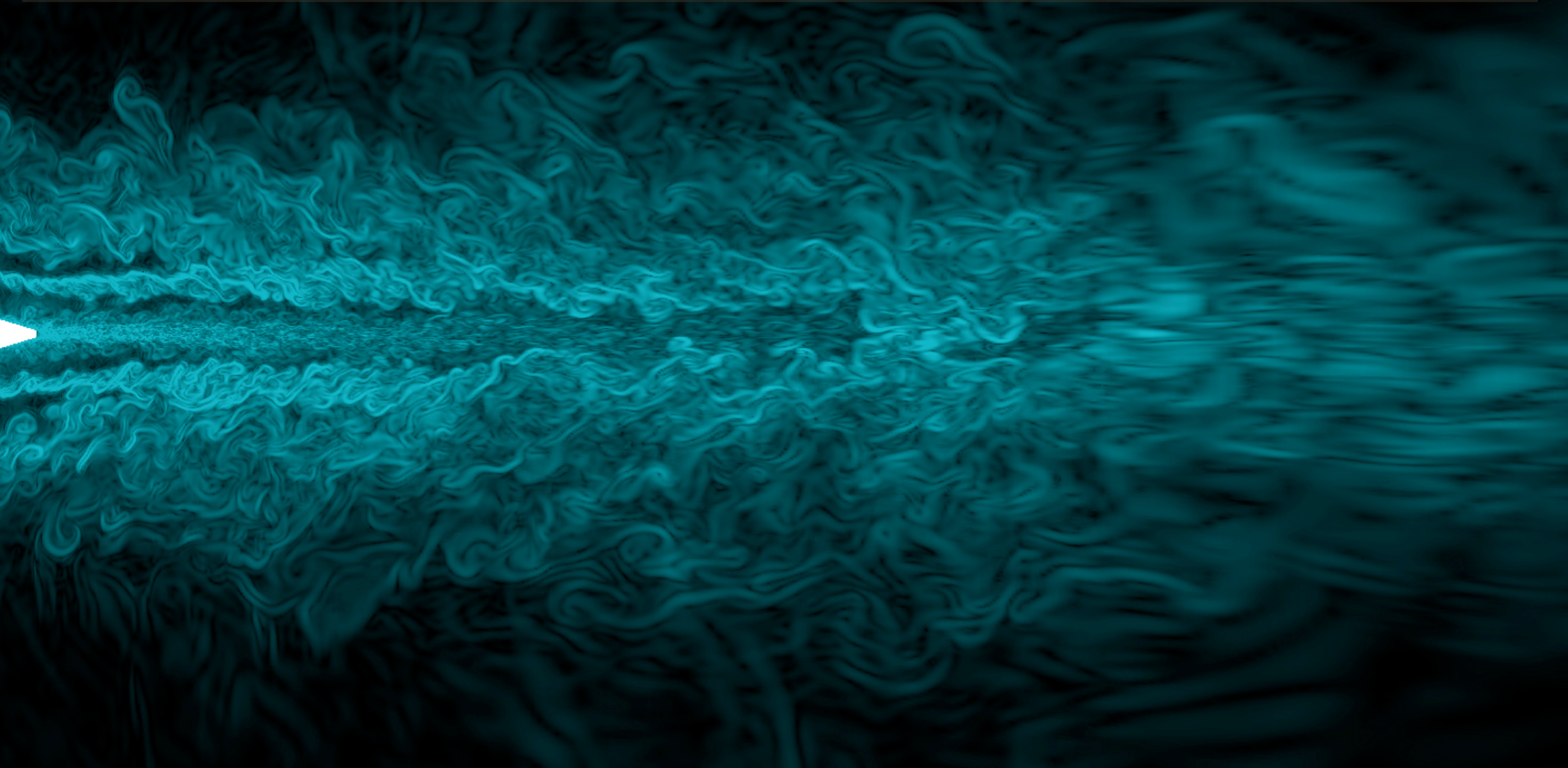


Imaging

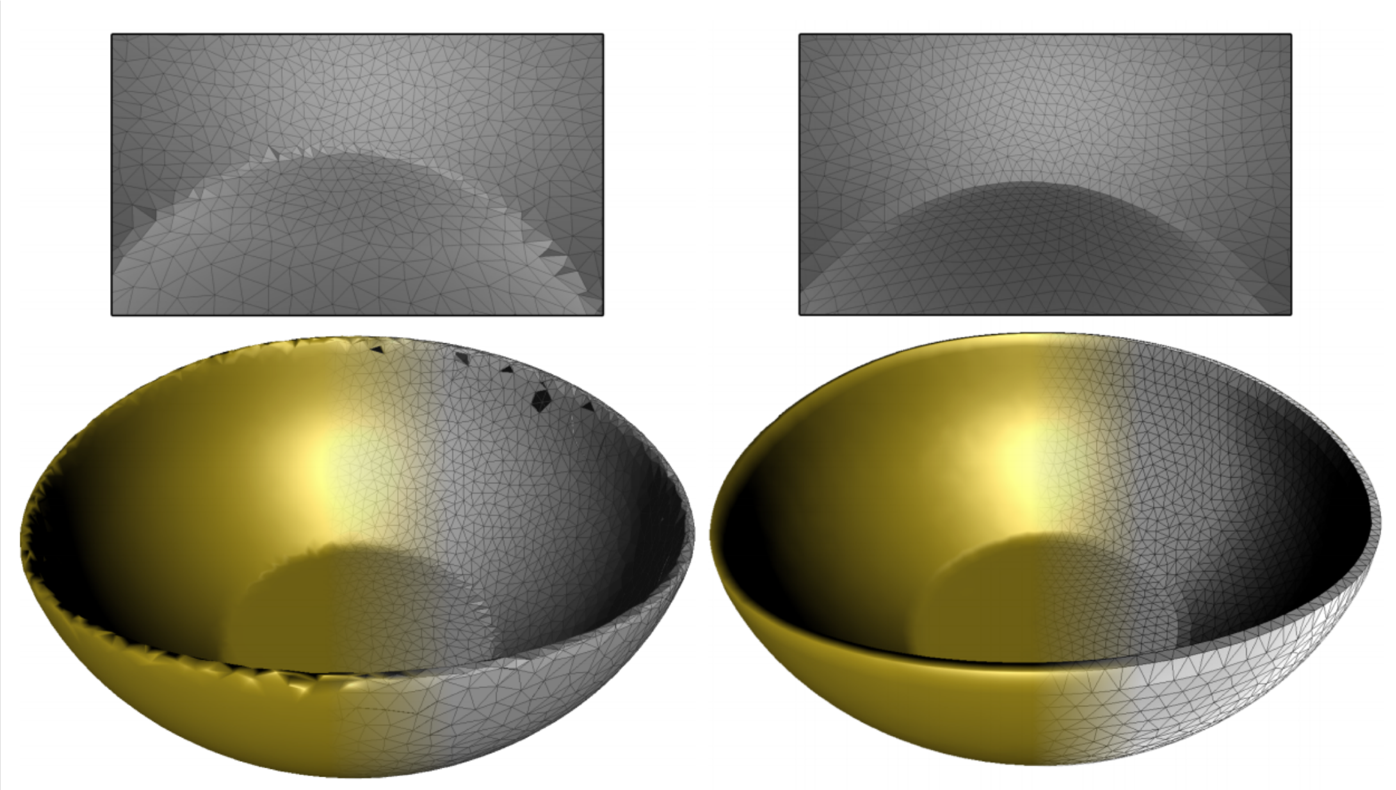
Patterns of Nature

Dynamic Structures

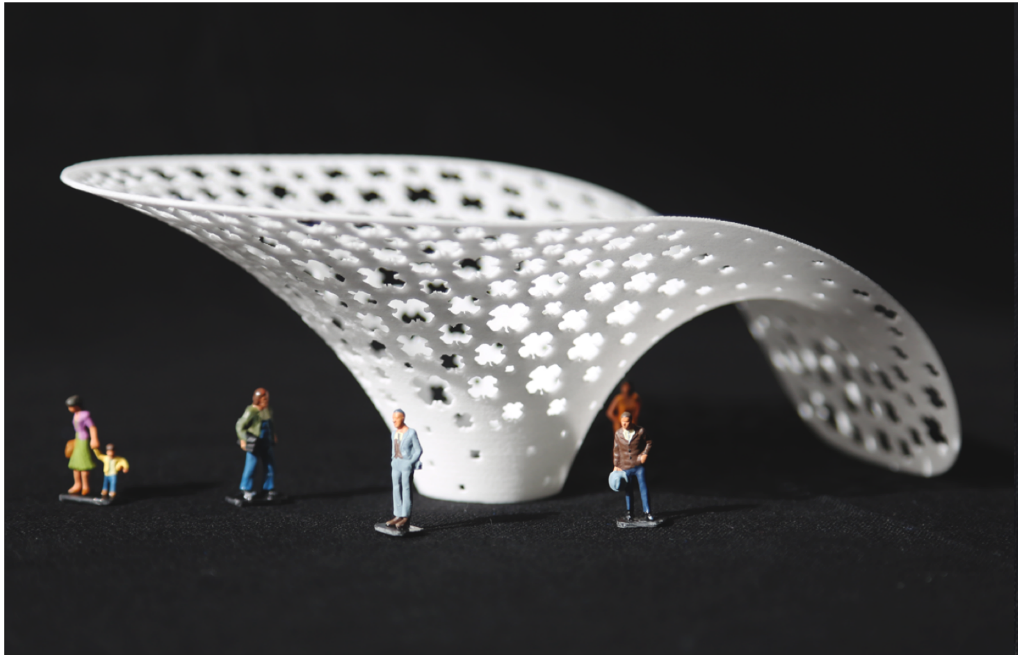
Simulations

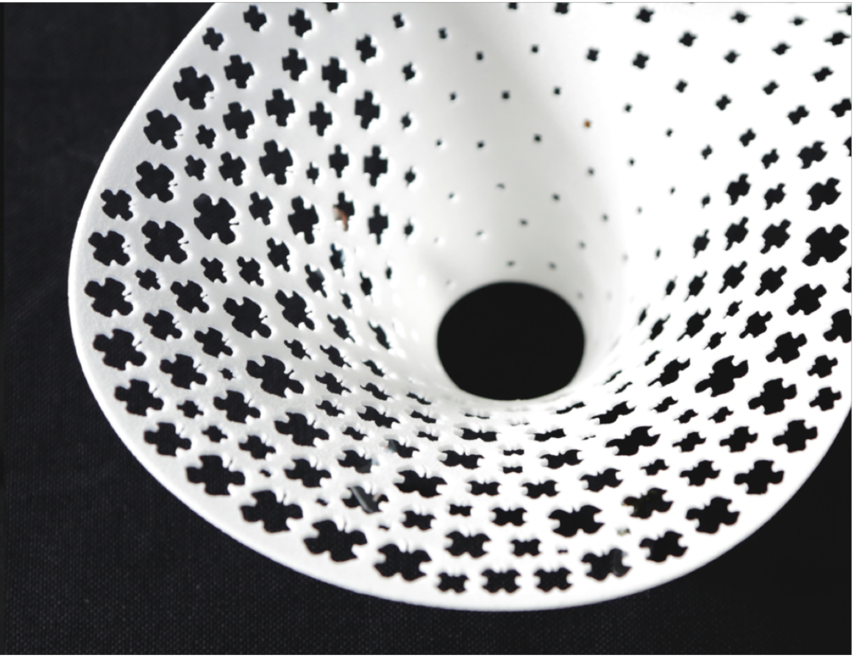


Geometry Processing

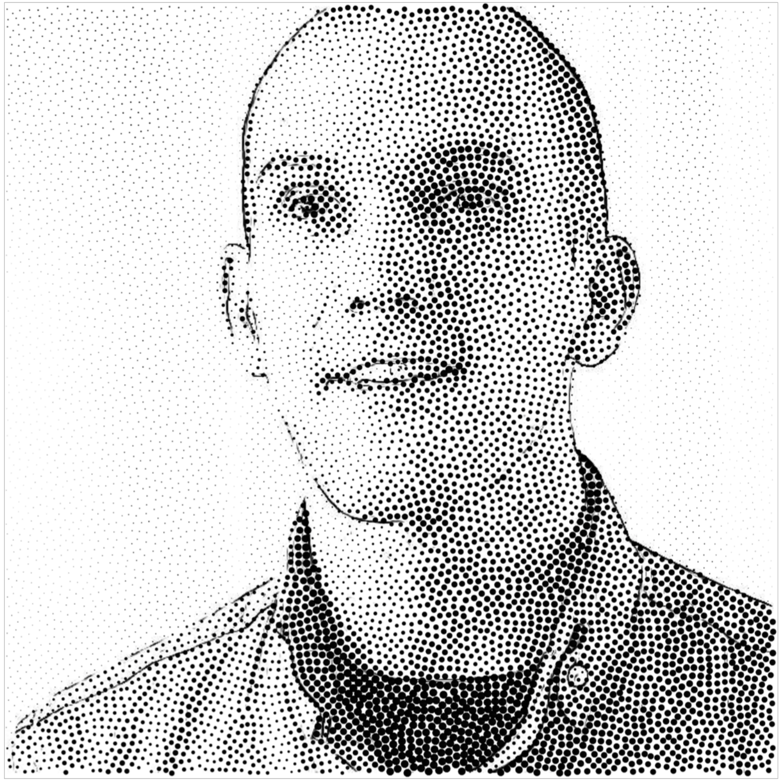


Fabrication

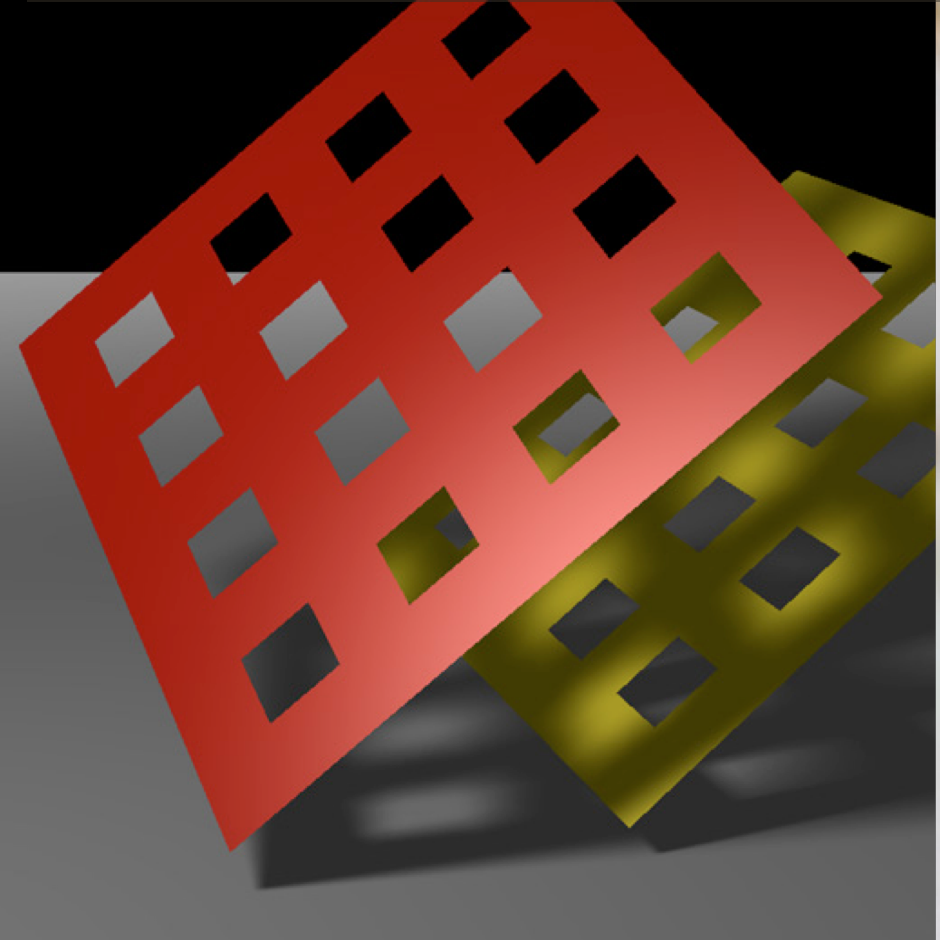




Non-photorealistic Rendering



Rendering – Computing Integrals



Estimating Integrals with Points

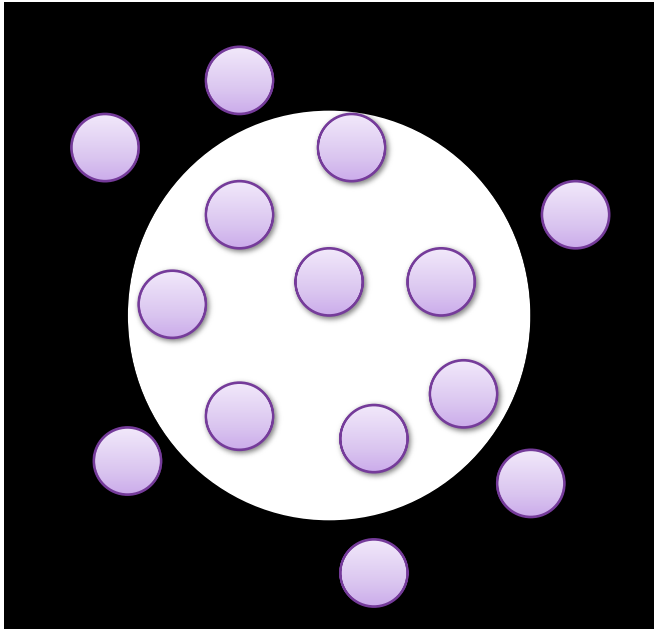
Sample and sum the sampled values of an integrand

$$I := \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} f(\mathbf{x})d\mathbf{x}$$

$$\hat{I} := \sum_{I=1}^{n} w_i f(\mathbf{x}_i)$$

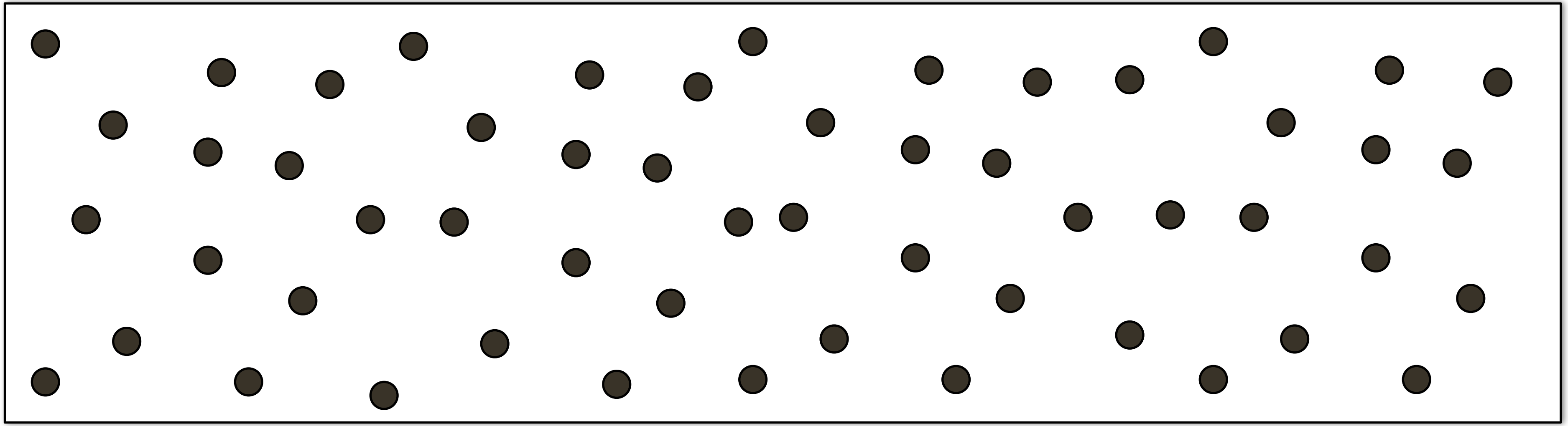
$$bias_{\mathcal{P}}[\hat{I}] = I - \mathbb{E}_{\mathcal{P}}[\hat{I}]$$

$$var_{\mathcal{P}}[\hat{I}] = \mathbb{E}_{\mathcal{P}}[\hat{I}^2] - (\mathbb{E}_{\mathcal{P}}[\hat{I}])^2$$



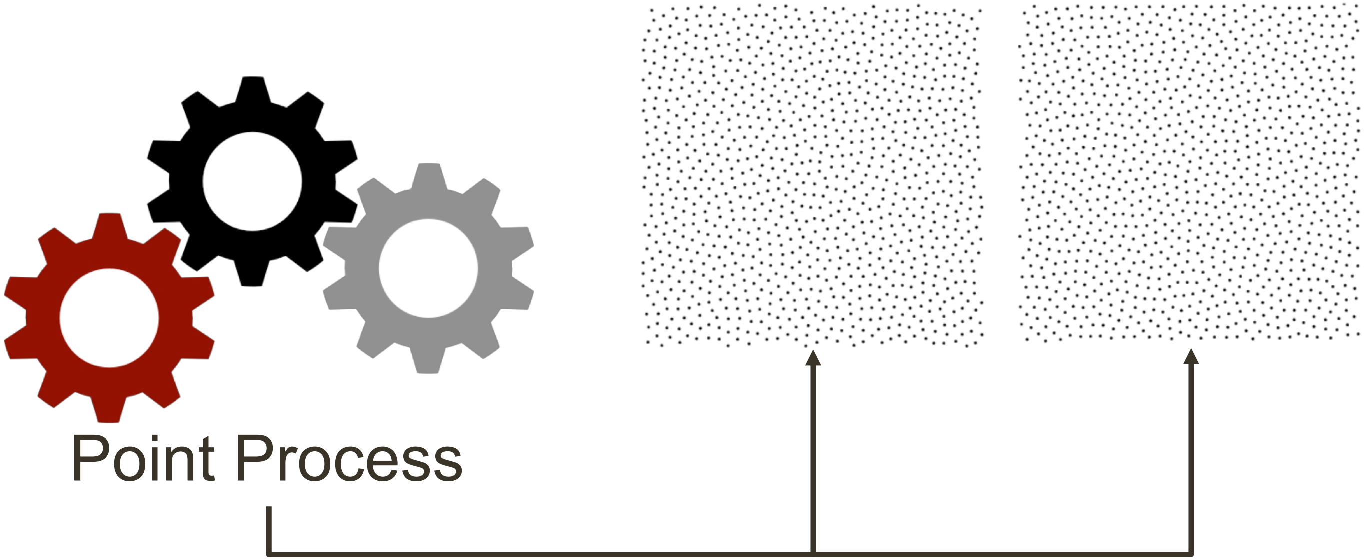
Stochastic Point Processes

Formal characterization of point patterns



Stochastic Point Processes

Formal characterization of point patterns



Stochastic Point Processes

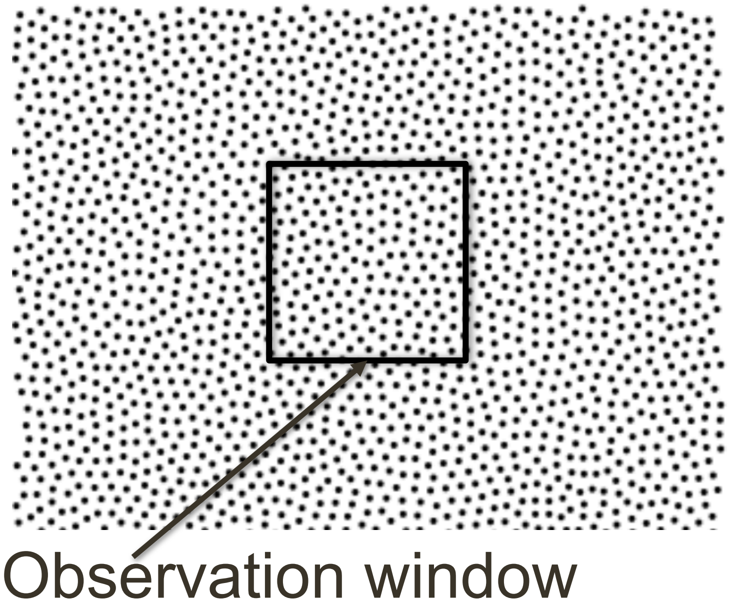
Examples of point processes

Natural Process

Manuel Process

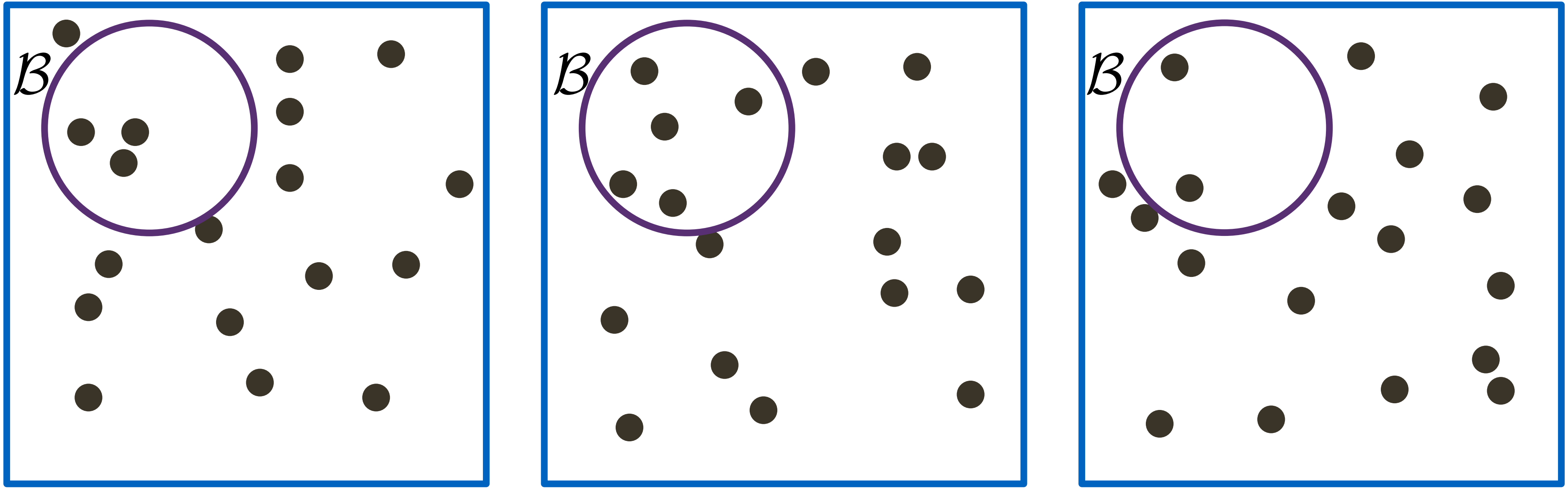
General Point Processes

Infinite point processes



General Point Processes

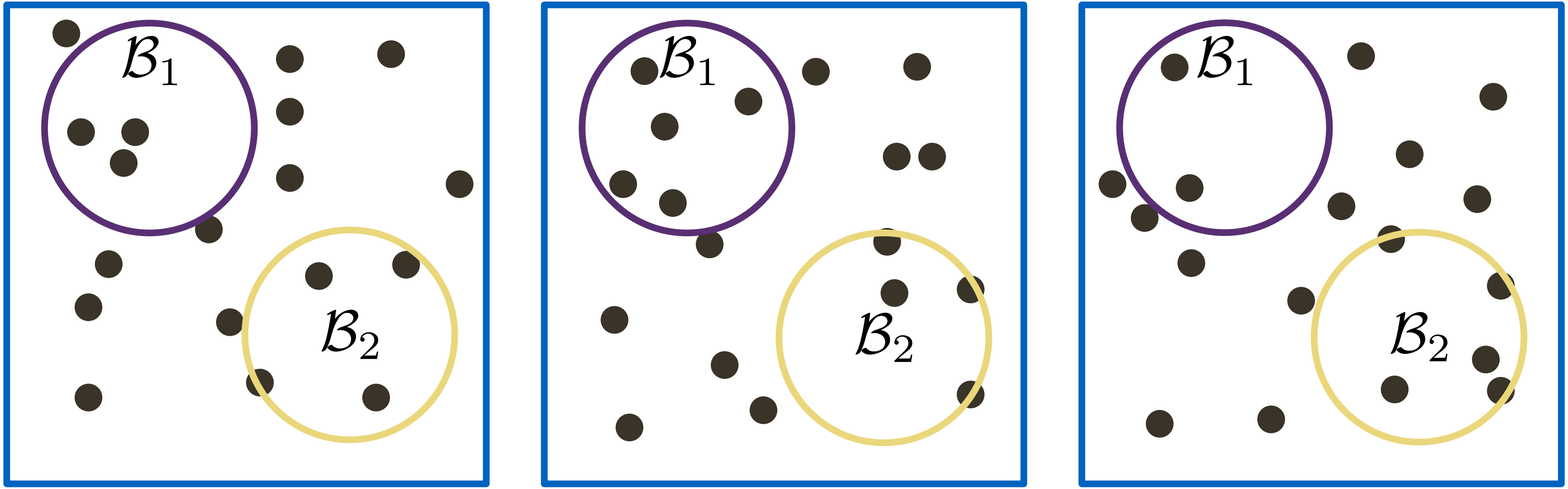
Assign a random variable to each set



$N(\mathcal{B}) = 3$ $N(\mathcal{B}) = 5$ $N(\mathcal{B}) = 2$

General Point Processes

Joint probabilities define the point process



$$p_{N(\mathcal{B}_1),N(\mathcal{B}_2)}$$

Point Process Statistics

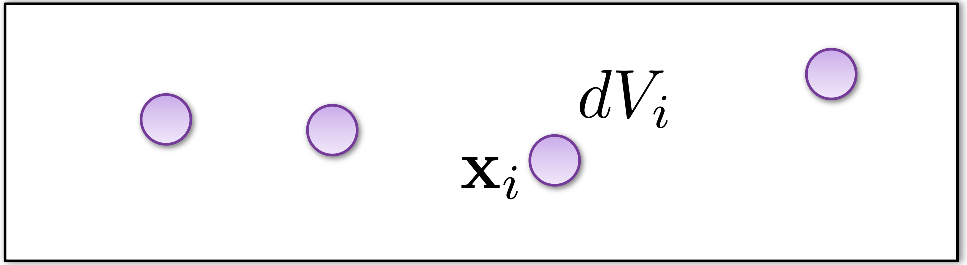
Correlations as probabilities

$$\varrho^{(n)}(\mathbf{x}_1, \cdots, \mathbf{x}_n) dV_1 \cdots dV_n = p(\mathbf{x}_1, \cdots, \mathbf{x}_n)$$

Product density

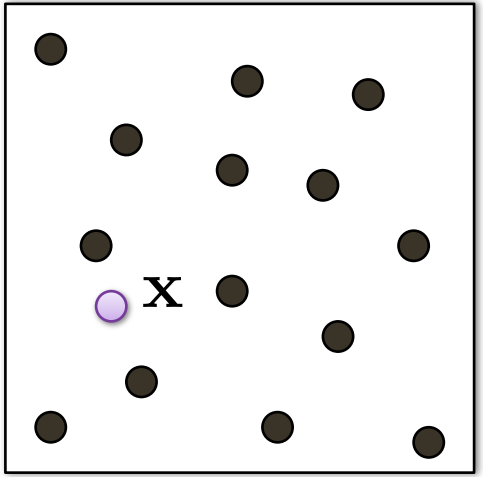
Small volumes

Points in space



Point Process Statistics

First order product density



$$\varrho^{(1)}(\mathbf{x}) = \lambda(\mathbf{x})$$

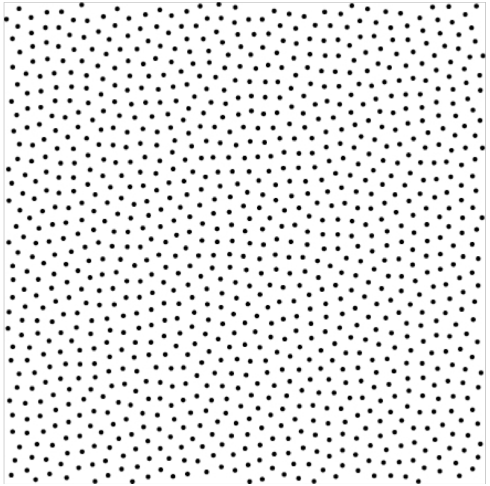
Expected number of points around **x**

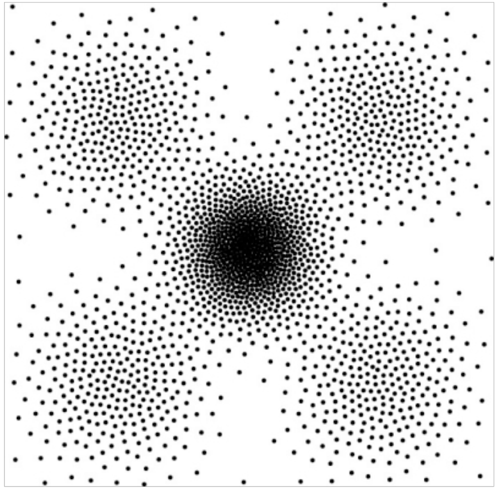
Measures local density

Point Process Statistics

First order product density

$\lambda(\mathbf{x})$



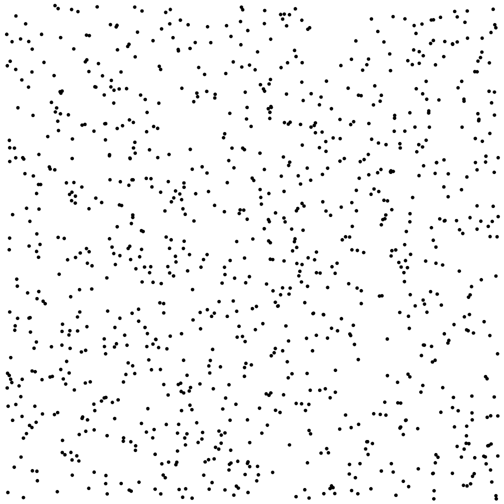


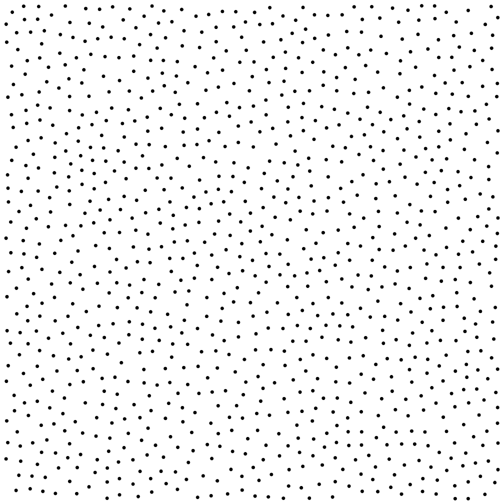
Point Process Statistics

First order product density

$\lambda(\mathbf{x})$

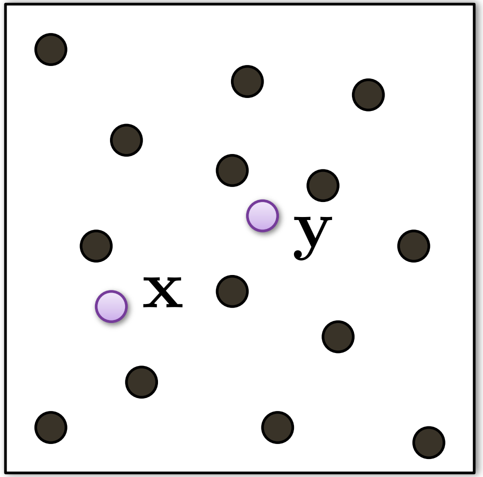
Constant





Point Process Statistics

Second order product density



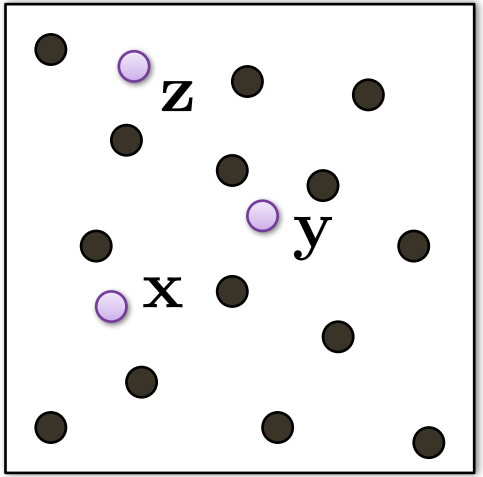
$$\varrho^{(2)}(\mathbf{x},\mathbf{y}) = \varrho(\mathbf{x},\mathbf{y})$$

Expected number of points around **x** & **y**

Measures the joint probability $p(\mathbf{x},\mathbf{y})$

Point Process Statistics

Higher order product density?



Expected number of points around x, y, z

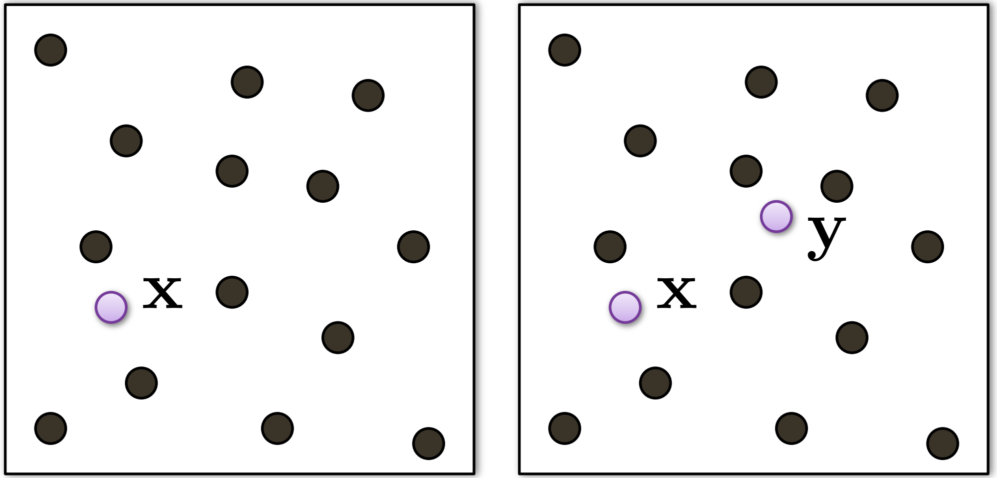
Not necessary: second order dogma

Point Process Statistics

Higher order not necessary: *second order dogma*

$$\varrho^{(1)}(\mathbf{x}) = \lambda(\mathbf{x})$$

$$\varrho^{(2)}(\mathbf{x}, \mathbf{y}) = \varrho(\mathbf{x}, \mathbf{y})$$

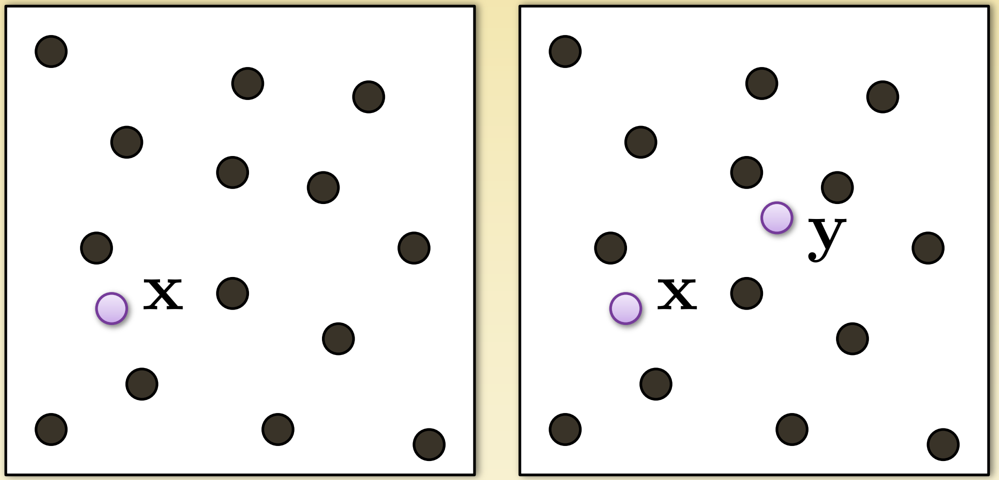


Point Process Statistics

Summary: 1st & 2nd order correlations sufficient

$$\varrho^{(1)}(\mathbf{x}) = \lambda(\mathbf{x})$$

$$\varrho^{(2)}(\mathbf{x}, \mathbf{y}) = \varrho(\mathbf{x}, \mathbf{y})$$



Point Process Statistics

Example: homogenous Poisson point process
a.k.a. random sampling

$$p(\mathbf{x}) = p$$

$$\lambda(\mathbf{x}) dV = p$$
$$\lambda(\mathbf{x}) = \lambda$$

$$p(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}) p(\mathbf{y})$$

$$\begin{aligned} p(\mathbf{x}, \mathbf{y}) &= \varrho(\mathbf{x}, \mathbf{y}) dV_x dV_y \\ &= p(\mathbf{x}) p(\mathbf{y}) \\ &= \lambda(\mathbf{x}) dV_x \lambda(\mathbf{y}) dV_y \end{aligned}$$

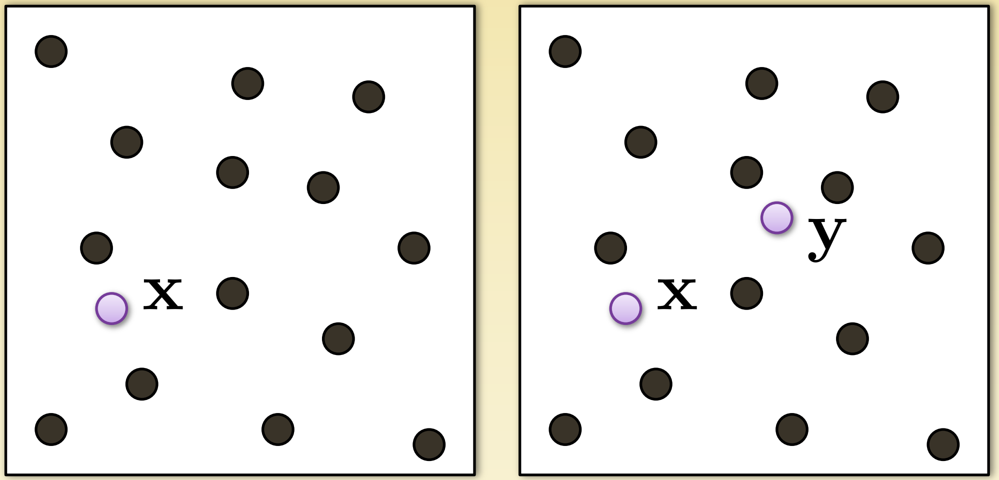
$$\varrho(\mathbf{x}, \mathbf{y}) = \lambda(\mathbf{x}) \lambda(\mathbf{y}) = \lambda^2$$

Point Process Statistics

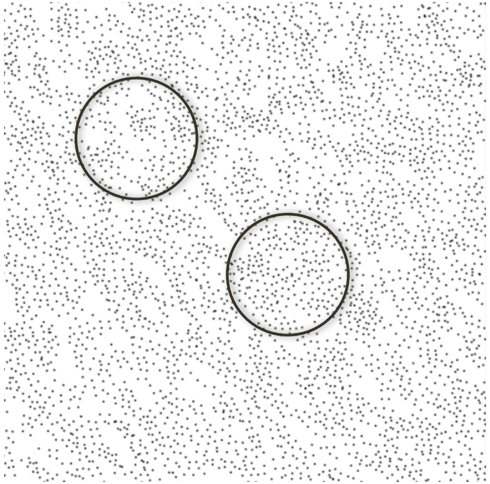
Summary: 1st & 2nd order correlations sufficient

$$\varrho^{(1)}(\mathbf{x}) = \lambda(\mathbf{x})$$

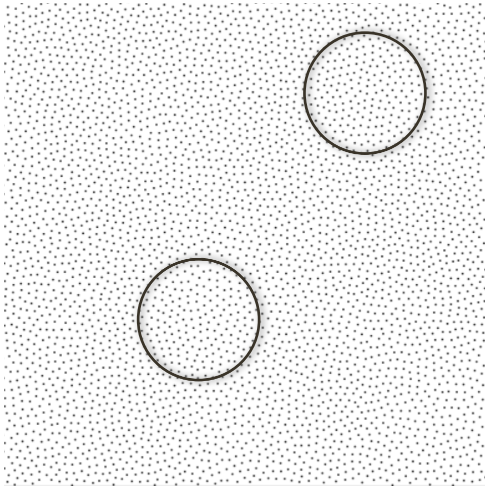
$$\varrho^{(2)}(\mathbf{x}, \mathbf{y}) = \varrho(\mathbf{x}, \mathbf{y})$$



Stationary Point Processes



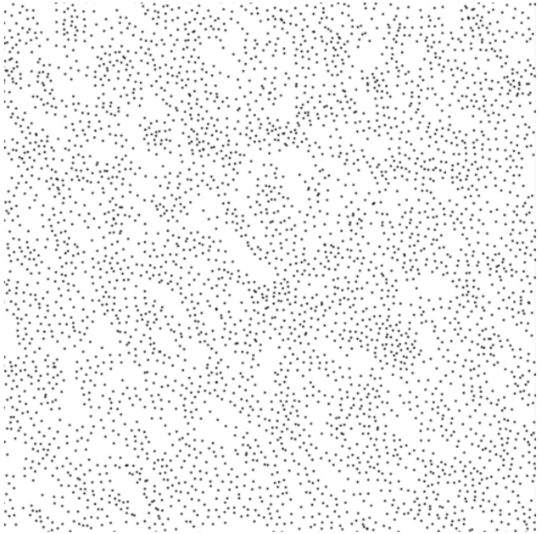
Stationary
(translation invariant)



Isotropic
(translation & rotation invariant)

Stationary Point Processes

Stationary (translation invariant)



$$\lambda(\mathbf{x}) = \lambda$$

$$\varrho(\mathbf{x}, \mathbf{y}) = \varrho(\mathbf{x} - \mathbf{y})$$

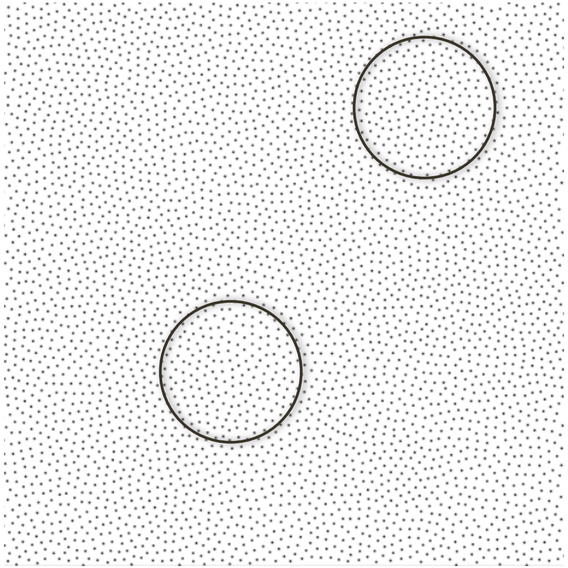
$$= \lambda^2 g(\mathbf{x} - \mathbf{y})$$

Pair Correlation Function (PCF)

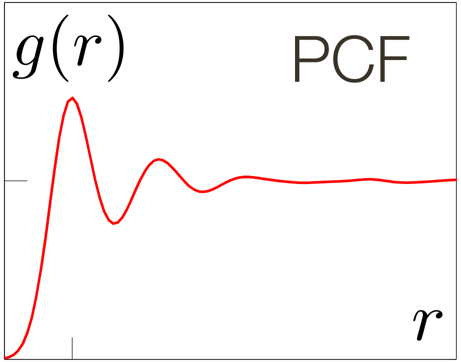
DoF reduced from 2d to d

Stationary Point Processes

Isotropic point process (translation & rotation invariant)



$$\lambda(\mathbf{x}) = \lambda$$
$$g(\mathbf{x} - \mathbf{y}) = g(||\mathbf{x} - \mathbf{y}||)$$



Estimating Correlations

Campbell's Theorem

$$\mathbb{E}_{\mathcal{P}}\left[\sum f(\mathbf{x}_i)\right] = \int_{\mathbb{R}^d} f(\mathbf{x})\boxed{\lambda(\mathbf{x})}d\mathbf{x}$$

$$\mathbb{E}_{\mathcal{P}}\left[\sum_{i \neq j} f(\mathbf{x}_i, \mathbf{x}_j)\right] = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(\mathbf{x}, \mathbf{y})\boxed{\varrho(\mathbf{x}, \mathbf{y})}d\mathbf{x}d\mathbf{y}$$

Estimating Correlations

First order $\lambda(\mathbf{x})$

$$\mathbb{E}_{\mathcal{P}}\left[\sum \mathbb{I}_{\mathcal{D}}(\mathbf{x}_i)\right] = \mathbb{E}_{\mathcal{P}}\left[\sum_{\mathbf{x}_i \in \mathcal{D}} 1\right]$$

$$= \int_{\mathcal{D}} \lambda d\mathbf{x} = \lambda \int_{\mathcal{D}} d\mathbf{x} = \lambda|\mathcal{D}|$$

Point distribution

Number of point distributions

$$\hat{\lambda} = \frac{\sum_{\mathcal{P}_k} N_k(\mathcal{D})}{K|\mathcal{D}|}$$

Estimating Correlations

Second order stationary - pair correlation function (PCF)

$$\mathbb{E}_{\mathcal{P}}\left[\sum_{i\neq j}\delta(\mathbf{r}-(\mathbf{x}_i-\mathbf{x}_j))\right]$$

$$=\int_{\mathbb{R}^d\times\mathbb{R}^d}\delta(\mathbf{r}-(\mathbf{x}-\mathbf{y}))\varrho(\mathbf{x}-\mathbf{y})d\mathbf{x}d\mathbf{y}$$

$$=\lambda^2\int_{\mathbb{R}^d\times\mathbb{R}^d}\delta(\mathbf{r}-(\mathbf{x}-\mathbf{y}))g(\mathbf{x}-\mathbf{y})d\mathbf{x}d\mathbf{y}=\lambda^2 g(\mathbf{r})$$

Estimating Correlations

Second order stationary - pair correlation function (PCF)

$$\hat{g}(\mathbf{r}) = \frac{1}{K\lambda^2} \sum_{\mathcal{P}_k} \sum_{\mathbf{x}_i,\mathbf{x}_j \in \mathcal{P}_k, i \neq j} \delta(\mathbf{r} - (\mathbf{x}_i - \mathbf{x}_j))$$

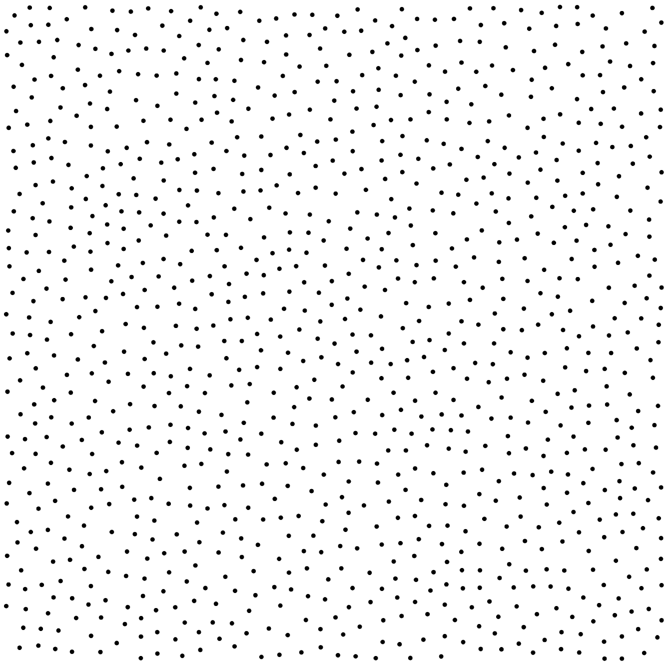
Finite domains:

$$\hat{g}(\mathbf{r}) = \frac{1}{K\lambda^2 a_{\mathbb{I}_\mathcal{D}}(\mathbf{r})} \sum_{\mathcal{P}_k} \sum_{\mathbf{x}_i,\mathbf{x}_j \in \mathcal{P}_k, i \neq j} \delta(\mathbf{r} - (\mathbf{x}_i - \mathbf{x}_j))$$

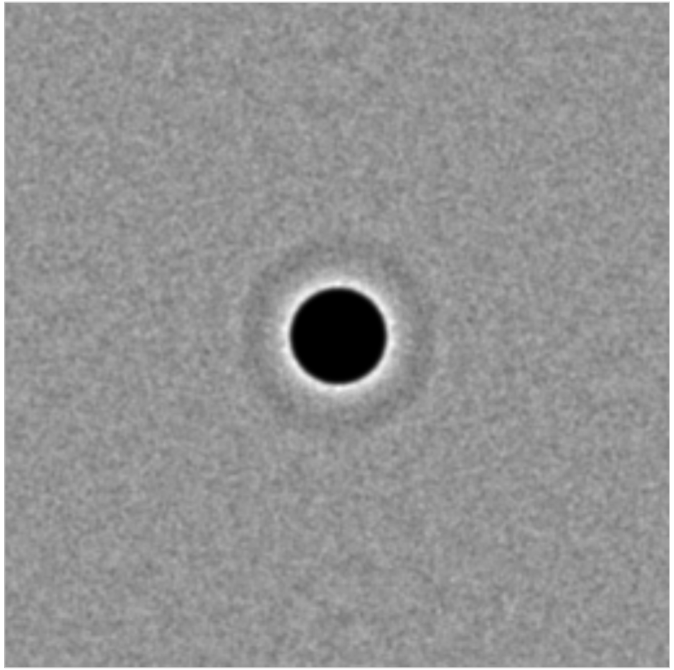
Estimating Correlations

Second order stationary - pair correlation function (PCF)

Point Distribution



Pair Correlation Function



Estimating Correlations

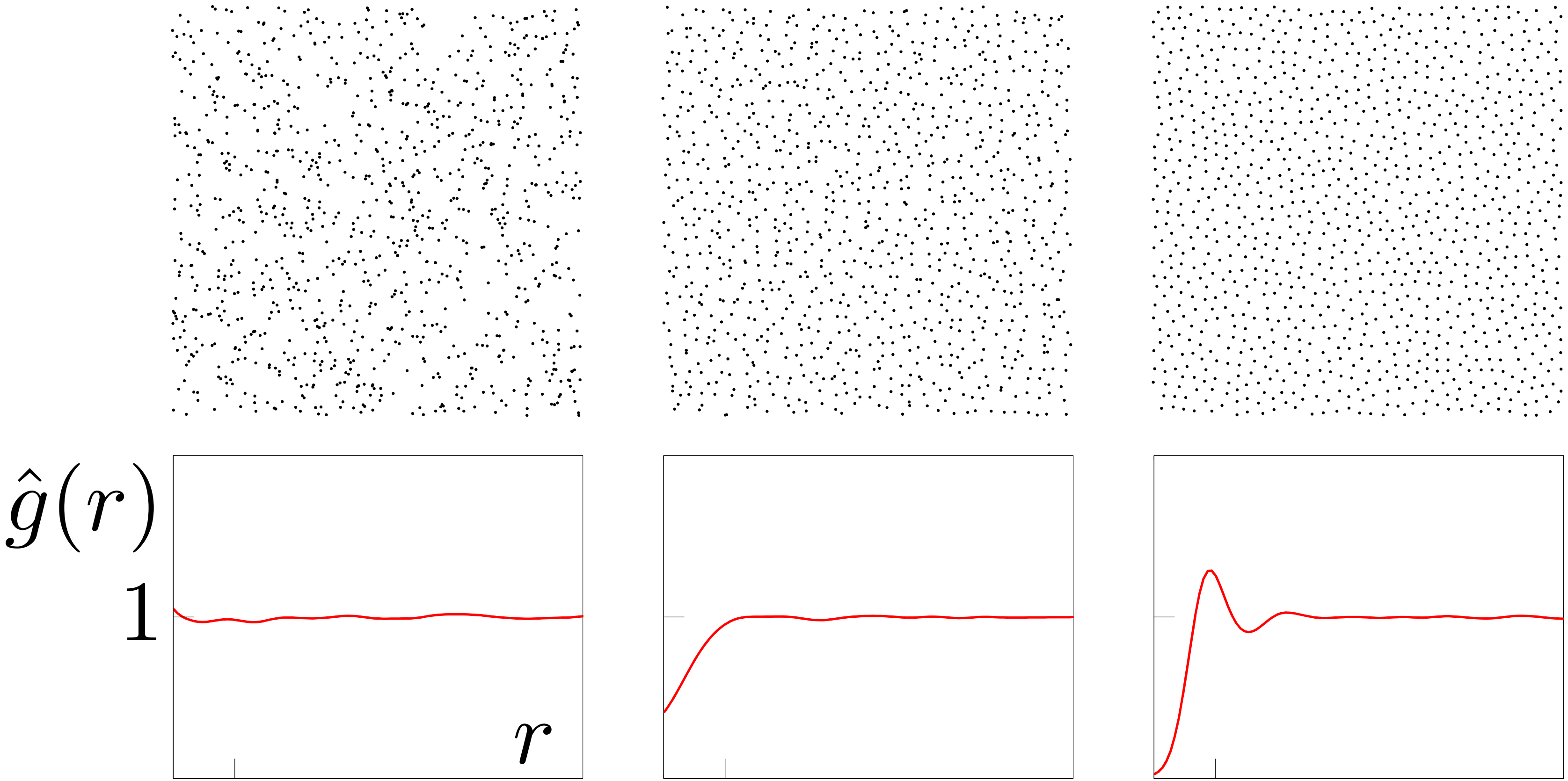
Second order isotropic - pair correlation function (PCF)

$$\hat{g}(r) = \frac{1}{\lambda^2 r^{d-1} |\mathcal{S}_d|} \sum_{i \neq j} k(r - \|\mathbf{x}_i - \mathbf{x}_j\|)$$

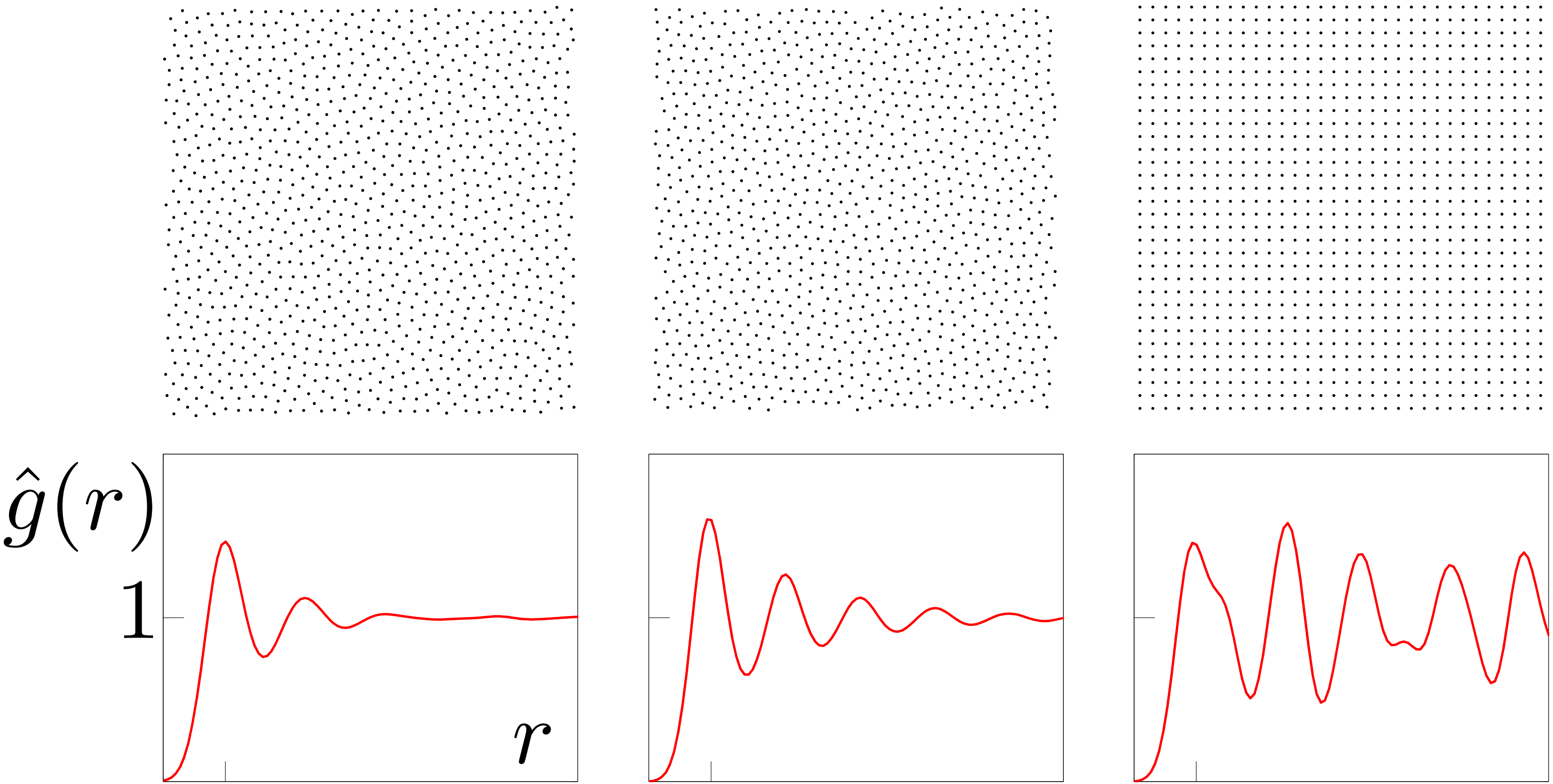
Volume of the unit hypercube in d dimensions

Kernel
e.g. Gaussian

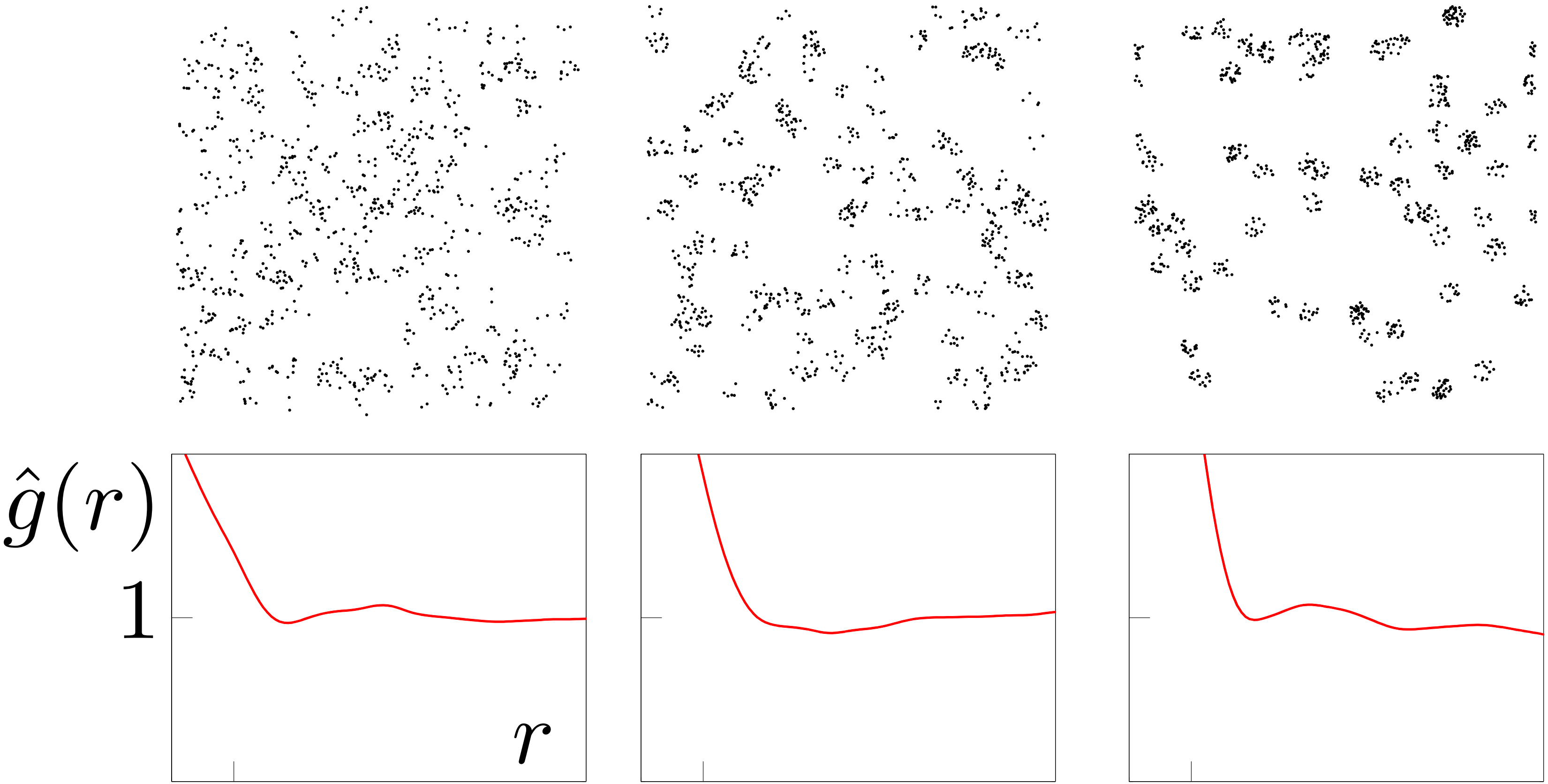
Pair Correlation Function



Pair Correlation Function



Pair Correlation Function



Spectral Statistics

$$P(\boldsymbol{\nu}) = \lambda G(\boldsymbol{\nu}) + 1$$

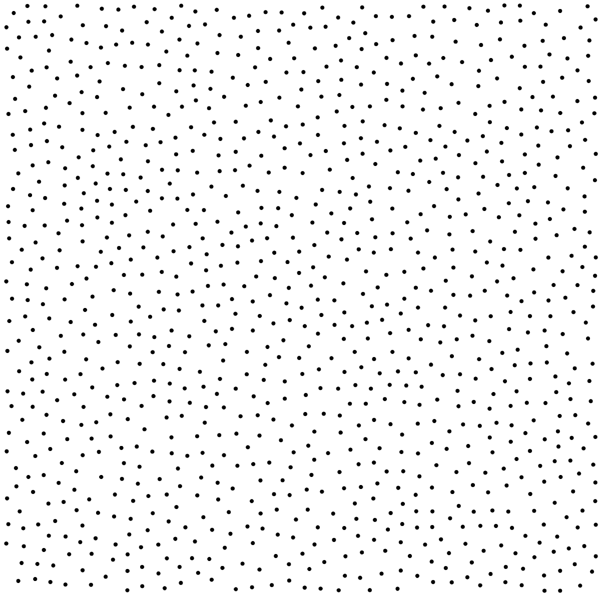
Power spectrum

Fourier transform of PCF

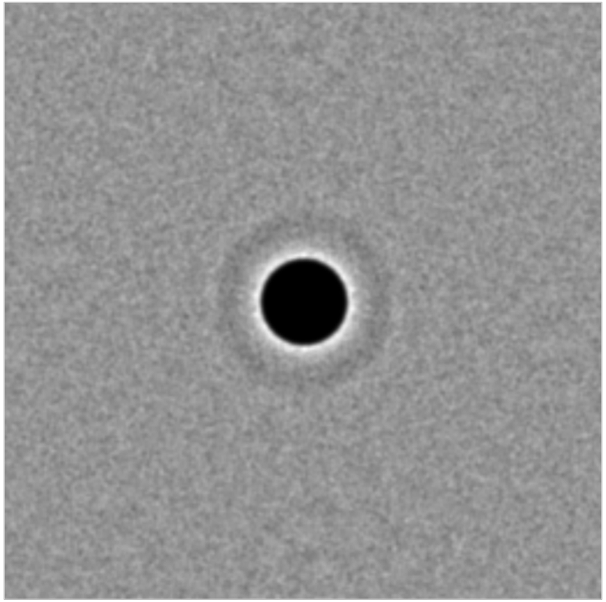
Spectral Statistics

$$P(\boldsymbol{\nu}) = \lambda G(\boldsymbol{\nu}) + 1$$

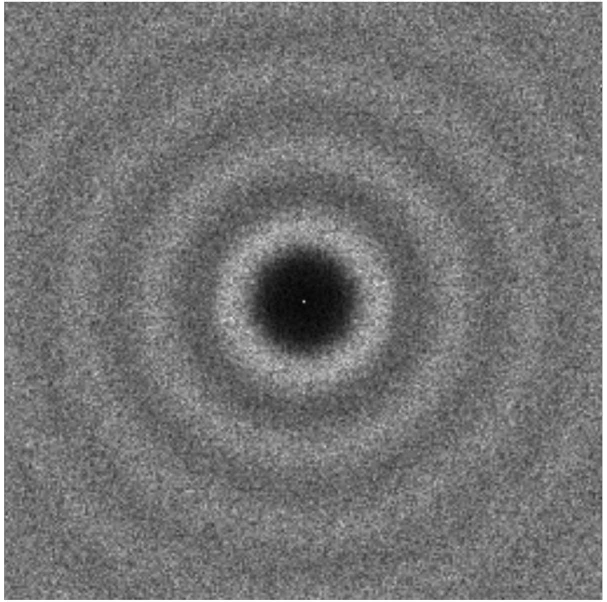
Points



PCF



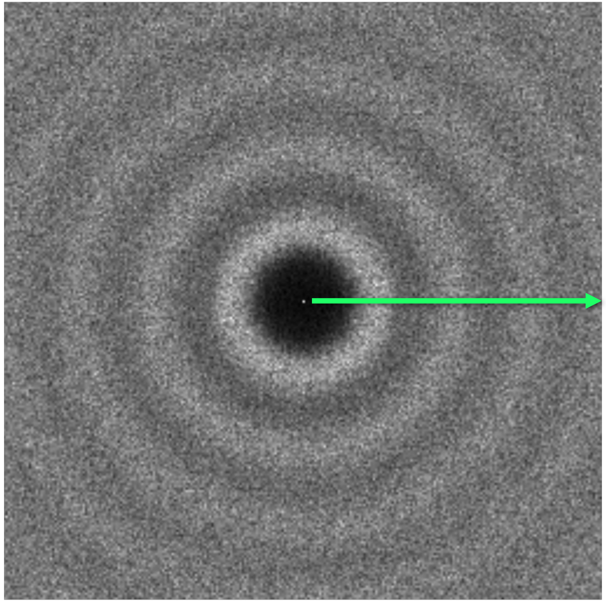
Power spectrum



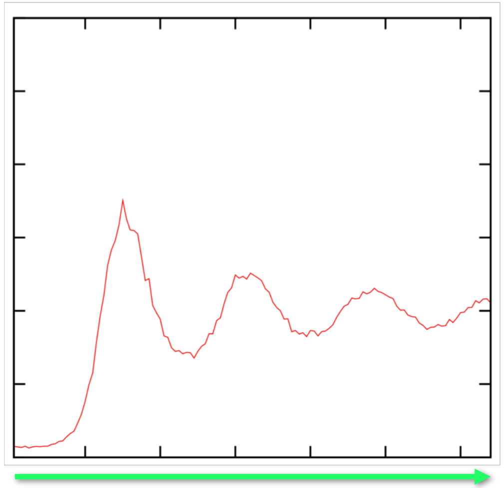
Spectral Statistics

$$P(\boldsymbol{\nu}) = \lambda G(\boldsymbol{\nu}) + 1$$

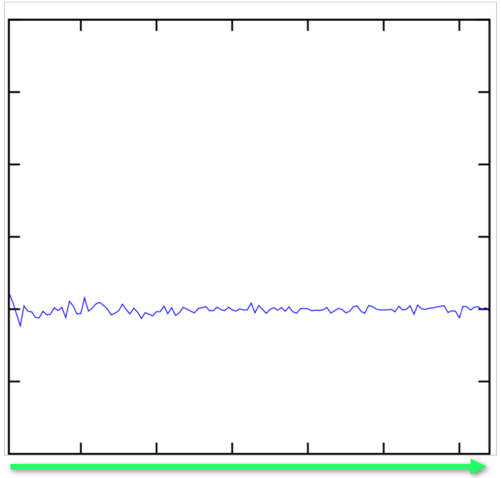
Power spectrum



Radial average



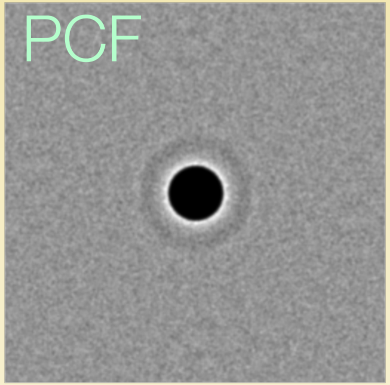
Radial anisotropy

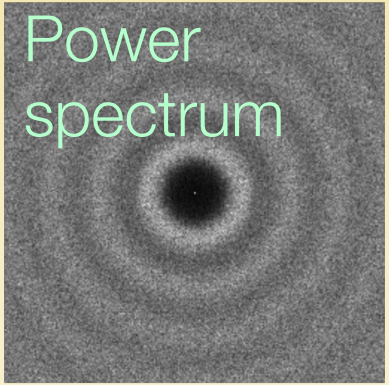


Statistics for Stationary Processes

Summary

Stationary: Spatial (PCF) & spectral (power spectrum)





Isotropic: radial averages